PARTIAL DIFFERENTIAL EQUATIONS

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5. Nonlinear elliptic PDE and the Calculus of Variations

(1) Prove Jensen's inequality: if $\phi : \mathbb{R} \to \mathbb{R}$ is smooth and convex, then for any $f \in C(\Omega)$ we have

$$\phi\left(\int_\Omega f d\mu\right) \le \int_\Omega \phi(f) d\mu,$$

provided that the measure μ satisfies $\int_\Omega d\mu = 1$.

(2 points)

(2) Prove that, in dimension 1, there is no function $u \in C^2([-1,1])$ that minimizes the functional

$$\int_{-1}^{1} x^2 |u'(x)|^2 dx$$

among all C^2 functions satisfying $u(-1) = -1$ and $u(1) = 1$.

(3 points)

(3) Let $L(p, u, x)$ be smooth and uniformly convex in p, and let $u \in C^2(\Omega)$ be a minimizer of

$$\int_\Omega L(\nabla u, u, x)$$

among all functions satisfying $u = g$ on $\partial\Omega$.

Find the PDE satisfied by u inside Ω.

(3 points)

(4) Let $\Gamma = \{(x, y) \in \Omega \times \mathbb{R} : y = u(x)\} \subset \mathbb{R}^{n+1}$ be a smooth hypersurface, given as the graph of a function $u \in C^\infty(\Omega)$. Prove that the mean curvature of Γ at a point $x \in \Omega$ is given by

$$H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)$$

(3 points)

(5) Assume that $u \in C^\infty(\overline{\Omega})$ is a minimizer of the area functional

$$\int_\Omega \sqrt{1 + |\nabla w|^2}\, dx,$$

among all functions with fixed boundary conditions $w = g$ on $\partial\Omega$, and with fixed volume

$$\int_\Omega w = 1.$$

Prove that the graph of u is a hypersurface of constant mean curvature.

Hint: The mean curvature is given by $H := \operatorname{div}(\nabla u/\sqrt{1+|\nabla u|^2})$

(3 points)

(6) Given $\phi \in C^\infty(\overline{\Omega})$, find a functional $\int_\Omega L(\nabla u, x)$ so that its corresponding Euler-Lagrange equation is the PDE

$$-\Delta u + \nabla\phi \cdot \nabla u = 0 \quad \text{in} \quad \Omega.$$

Hint: Try functionals with an exponential term.

(3 points)

(7) Let $u \in C^\infty(\overline{\Omega})$ be a local minimizer of the functional

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(u)$$

with fixed boundary condition $u = g$ on $\partial\Omega$, that is, there exists $\delta > 0$ for which

$$\mathcal{E}(u+\xi) \geq \mathcal{E}(u) \qquad \forall \xi \in C_c^\infty(\Omega) \quad \text{with} \quad \|\xi\|_{L^\infty(\Omega)} \leq \delta.$$

Compute a "second derivative" of the functional to deduce that

$$\int_\Omega |\nabla\eta|^2 \geq \int_\Omega f'(u)\eta^2 \quad \text{for every} \quad \eta \in C_c^\infty(\Omega).$$

Note: This inequality is satisfied by local minimizers of the functional (and asymptotically stable solutions of the corresponding parabolic PDE). However, it does not hold in general for all solutions of the corresponding PDE.

(4 points)

(8) Given $\varphi \in C_c^\infty(\Omega)$, prove that there exists a function u that minimizes the Dirichlet integral among all functions $w \in H_0^1(\Omega)$ satisfying $w \geq \varphi$ in Ω.

(3 points)

(9) Let $n \geq 3$. Prove that $u(x) = \log\frac{1}{|x|^2}$ belongs to $H^1(B_1)$ and is a solution of

$$\begin{cases} -\Delta u &= \kappa_n e^u & \text{in} \quad B_1 \\ u &= 0 & \text{on} \quad \partial B_1, \end{cases}$$

for some constant $\kappa_n > 0$.

This shows that solutions to nonlinear PDE can be singular, i.e., with $u \to \infty$ at an interior point.

(2 points)

(10) Find a positive solution of

$$-\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in} \quad B_1$$

of the form $u(x) = a/(1-|x|^2)^\beta$ for positive constants a, β.

This shows that solutions to nonlinear PDE can be smooth inside a domain and yet become infinity *everywhere* on its boundary.

(2 points)

(11) Let $u \in C^2(\overline{\Omega})$ be a minimizer of the functional

$$\mathcal{E}(w) := \int_\Omega \tfrac{1}{2}|\nabla w|^2$$

over all functions satisfying

$$\int_\Omega G(w)dx = 1.$$

Show that u solves

$$\begin{cases} -\Delta u = \lambda g(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

for some constant λ, where $G' = g$.

(3 points)

(12) Prove that for any $\alpha \in (0,1)$ there exists a minimizer $u \in H_0^1(\Omega)$ of the functional

$$\mathcal{E}(w) := \int_\Omega \tfrac{1}{2}|\nabla w|^2$$

over all functions $w \in H_0^1(\Omega)$ satisfying

$$\int_\Omega |w|^{\alpha+1}dx = 1.$$

(3 points)